EXPLORING THE RELATIONSHIP BETWEEN EXPLANATIONS AND EXAMPLES: PARITY AND EQUIVALENT FRACTIONS

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Examples and explanations are inherent elements of mathematics learning and teaching. This study explores the relationship between examples and explanations given for the same concept. Results indicated that for the concept of parity, fifth grade students offer different explanations for different examples. However, for the concept of equivalent fractions, algorithmic explanations were most preferred.

INTRODUCTION

Mathematical concepts are complex and multi-faceted. That is, there may be different equivalent ways of defining a concept, but they all necessarily lead to one set of critical attributes. While mathematically, all of the critical attributes of a specific concept are equally essential and should be equally attributed to that concept, psychologically, they may not be the same. On the one hand, students often associate non-critical attributes with some concept; on the other hand, students may associate a concept with a shorter list of critical attributes than it truly has (Hershkowitz, 1990). How can we help students attain a fuller range and more encompassing recognition of all critical attributes for a mathematical concept?

Several studies have focused on the roles of examples in concept formation and expanding a student's accessible example space (e.g., Watson & Mason, 2005). Yet, recognizing a wide set of examples as being instances of some concept, does not necessarily mean that the student will also recognize a wide set of critical attributes as belonging to that concept. Another seemingly separate line of research related to learning mathematical concepts, is research concerning explanations. Previous studies have focused on the types of explanations used by teachers and students (e. g., Bowers & Doerr, 2001) as well as the sociomathematical norms related to giving and evaluating mathematical explanations (Levenson, Tirosh, & Tsamir, 2009). While those studies are perhaps implicitly related to concept development, they do not focus specifically on how explanations may be related to examples and how this relationship may inform us of students' conceptualizations.

According to Watson and Chick (2011), in order for students to learn from examples and see what an example could be an "example of", a process must take place which includes seeking relations between elements of an example. This study explores the possibility that explanations can play a role in this process, that examples may help students "focus mindfully" on the examples (p. 285). It seeks to combine research related to examples with research related to explanations, and to explore the relationship between examples and explanations given for the same concept. Using

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two mathematical contexts, parity and equivalent fractions, the following questions are investigated: Do different examples give rise to different explanations, or are some explanations more prevalent than others, regardless of the different examples? Do students consistently give the same explanation for a concept despite being shown different examples of that concept or do they give different explanations for different examples?

EXAMPLES AND EXPLANATIONS

One key to using examples in concept formation may be variation. Zodik and Zaslavsky (2008) refer to Watson and Mason's (2006) discussion of variation in structuring tasks, and suggest a similar way of structuring examples. Just as some features of a task may vary while others are kept constant, so too with examples. The presentation of examples should be structured in such a way as to highlight relevant features while downplaying non-critical attributes. Rowland (2008) also claims that we learn from discerning variation and that "the provision of examples must therefore take into account the dimensions of variation inherent in the objects of attention" (p. 153). Learners need to be aware of which attributes of a concept can be varied and which cannot. At different times in the learning process, students may be aware of different dimensions which may be varied, sometimes claiming an unnecessarily restricted sense of possible variations (Goldeberg & Mason, 2008). In his study of examples in the teaching of elementary mathematics, Rowland (2008) found that some examples may obscure the role of variables, making it more difficult for students to learn from those examples. Recognizing the significance of teachers' choice of examples, Zodik and Zaslavsky (2008) investigated teachers' considerations in choosing examples. Their study pointed to several considerations, including choosing examples that will draw students' attention to relevant features.

Like examples, explanations are also used every day in the mathematics classroom and may have several functions. Explanations may answer a "how" question, and describe the procedure used to solve a problem, or they may answer a "why" question where the underlying assumption is that the explanation should rely on mathematical properties and generalizations (Levenson, Barkai, & Larson, 2013). Similarly, calculational explanations describe a process, procedure, or the steps taken to solve a problem. Conceptual explanations describe the reasons for the steps, which link procedures to the conceptual knowledge of the student (Bowers & Doerr, 2001). Hemmi, Lepik, and Viholainen (2013) inferred that explanations are an important part of problem solving and reasoning processes and may be used to make mathematical connections clear, even among young students. Explanations may also be given to rationalize actions, both for the giver of the explanation as well as for the receiver (Krummheuer, 2000). Nunokawa (2010) claimed that explanations not only communicate student's existing thoughts but may also generate new objects of thought by directing new explorations which may then deepen the student's understanding of the problem at hand. Thus, an underlying function of explanations is to expand students' mathematics learning.

METHODOLOGY

Tools and procedure

Two questionnaires, the parity questionnaire and the equivalent-fractions questionnaire, were handed out to 71 fifth-grade students. Both concepts had been introduced to students previously and it was expected that students would be familiar with both concepts at the time of the study. The parity questionnaire was handed out in the beginning of the year. On the parity questionnaire, students were asked to consider whether the integers 14, 9, 286, and 0 were even or odd and to explain their reasoning. The first two integers, 14 and 9, were chosen because they are relatively small natural numbers that elementary students can easily relate to, envision, and manipulate. By choosing one even and one odd number we could discern if the types of explanations given were related to the parity of the number. Zero was chosen because of the much researched difficulties, among students and teachers alike, conceptualizing and operating with this number (e.g., Anthony & Walshaw, 2004). The fourth number, 286, was chosen because it is not a number that is usually encountered by children in a day-to-day context and in order to see if there would be a difference between students' explanations for a two-digit number and a three-digit number. The fractions questionnaire was handed out approximately three months after the parity questionnaire. On the fractions questionnaire, students were asked to assess the equivalence of three pairs of fractions, 2/4 and 6/12, 5/15 and 10/30, 0/4 and 0/12, and to explain their reasoning. The first pair was chosen because it was thought that both fractions would be familiar to children, and like the numbers 14 and 9, they would be easy to relate to, envision, and manipulate. The second pair was chosen because they both reduce to one-third but students can also easily expand 5/15 to 10/30. The last pair was chosen because of its involvement with zero, as described above.

Analyzing the data

Students' assessments of the parity of the given numbers and of the equivalence of the given pairs of fractions were coded for correctness. Only explanations associated with correct responses were analysed further. On the parity questionnaire, four categories of explanations emerged from the data (See Table 1). Three explanations were mathematically-based (i.e., explanations based on mathematical definitions or previously learned mathematical properties, often using mathematical reasoning) and one was practically-based (PB). Some explanations were unequivocally wrong. For example, one student wrote that 9 is an odd number because "it is a prime number." Some explanations could not be categorized, such as "When you divide 14 by 7, nothing will stand by itself." These types of explanations were categorized as "other". On the fraction explanation, five categories emerged (see Table 2). As with the parity questionnaire some explanations (e.g., "2/4 is equivalent to 6/12 because all the numbers are even") were either invalid or incomprehensible and could not be categorized. Two experts in the field of mathematics education validated the categorization of explanations for each concept.

Theoretically, each type of explanation could be used for each example. For example, the parity of each number could be explained by claiming that it was or was not divisible by two. In order to assess students' tendencies to be consistent when explaining a concept, the explanations each student gave for the different examples were compared. For example, one student claimed that "14 is even because it's divisible by 2", "9 is not even because it is not divisible by 2", "0 is even because when you stand on 2 and jump backwards 2 steps you land on 0", and "286 is even because 6 is even." That student was consistent regarding the explanations given for parity of 14 and 9, but not consistent with regard to 14 and 0, and with regard to 14 and 286. If we consider all three even numbers together, we would say that the student was not consistent regarding the explanations given for even numbers. A similar comparison was carried out for the explanations given on the fractions questionnaire.

Categories	Students wrote…
Divisible by 2: An even number is divisible by 2, is a multiple of 2, or can be expressed as the sum of 2 equal whole numbers.	"14 can be written as 7 + 7", "14 is an even number because it's divisible by two without a remainder."
Number line: Even numbers are alternating whole numbers on the number line when you start with 0.	"When you start from 0 on the number line, jumping by twos, we end up standing on the number 14."
Last digit rule: If the ones digit of a number is even, then the whole number is even.	"14 is even because 4 is even."
Practically-based: Explanations that use daily contexts, drawing, and/or manipulatives.	"14 is even because if I want to give out 14 pencils to two children, each one would get the same amount of pencils."

Table 1: Categories of parity explanations

Categories	Students wrote…
Equal to the same number: Equivalent fractions are two fractions that represent the same number.	"5/15 and 10/30 are both equal to 1/3"
Algorithmic: Expansion or reduction of one or both fractions.	"If you multiply (both the numerator and denominator in 2/4) by 3, you get 6/12."
Numerator/denominator (N/D) relationship: The denominator is a multiple of the numerator.	"4 is twice as much as 2 and 12 is twice as much as 6."
Zero is nothing: relates zero to nothing.	"Because in both of them there is nothing."
Practically-based (PB): Explanations that use daily contexts, drawing, and/or manipulatives.	"Six and two are equal only with smaller pieces." (This explanation refers to the "pie" diagram where each "piece" is an equal fractional part of the whole pie.)

Table 2: Categories of equivalent fractions explanations

RESULTS

Parity questionnaire

As shown in Table 3, nearly all, of the students knew the parity of 14, 9, and 286. However, as expected, zero was a cause for confusion. Because one of the aims of this

study was to investigate the variability or, conversely, the consistency of explanations for different examples, only the explanations of students, who evaluated correctly the parity of all four integers, were examined. This led to a final sample of 56 students.

	14	9	0	286
Correct evaluations	71 (100)	70 (99)	57 (80)	68 (96)

Table 3: Frequencies (in %) of correct evaluations per integer (N=71)

Table 4 summarizes the results of how children explained the parity of each integer. Taking into consideration a total of 188 valid explanations, 35% of those explanations were based on divisibility by two, 34% were based on the number line, 29% were based on the last-digit rule, and 2% were practically-based. In other words, none of the categories stood out as being truly dominant over the others. On the other hand, for each integer, there seemed to be one type of explanation which was used more frequently than the others. For example, most students explained the parity of 14 by writing that it was divisible by two, while most students explained the parity of zero by its placement on the number line.

	Divisible by 2	Number line	Last-digit rule	PB	Other
14	23 (41)	11 (20)	16 (29)	-	6 (11)
9	22 (39)	17 (30)	2 (4)	2 (4)	13 (23)
0	8 (14)	31 (55)	2 (4)	1 (2)	14 (25)
286	13 (23)	5 (9)	35 (63)	-	3 (5)

Table 4: Frequencies (in %) of types of explanations per integer (N=56)

With regard to consistency, only valid comprehensible explanations were considered, leading to a sample of 42 students. Explanations for the two, relatively small and familiar integers, 14 and 9, were compared first. Following that comparison, explanations for each two even numbers were compared, and finally the explanations for all three even numbers were compared. The highest consistency of explanations occurred when explaining the parities of 14 and 9 and 14 and 286. While over a third of the students offered the same type of explanation for 14 and 0, only 19% of the students offered the same type of explanation for 0 and 286, leading to a relatively low consistency rate for all three even numbers.

Examples	14, 9	14, 0	14, 286	0, 286	14, 0, 286
Consistent explanations	27(64)	18(43)	25(60)	8(19)	7(17)

Table 5: Frequencies (in %) of consistent explanations for groups of integers (N=42)

Fractions questionnaire

Out of the 71 students who filled in the parity questionnaire, 66 students also filled in the fraction questionnaire. Results of students' assessments are summarized in Table 6. Once again, introducing zero into an example seemed to cause difficulties.

	2/4=6/12	5/15=10/30	0/4=0/12
Correct evaluations	65 (98)	63 (95)	48 (74)

Table 6: Frequencies (in %) of correct evaluations of equivalent fractions (N=66)

There were 48 students that knew that all three pairs of fractions were equivalent. Their explanations are summarized in Table 7. Of the 120 valid explanations, 48% were algorithmic, 31% related to the fractions being equal to the same number, 10% related to the relationship between the numerator and denominator (N/D), 7% were PB, and 4% considered the "zero is nothing" analogy. In other words, there seemed to be a clear preference for algorithmic explanations. When looking at the explanations given for the different examples, algorithmic explanations were most prevalent for the example 5/10=10/30. However, no clear preference of one type of explanation was found for the other examples. Interestingly, more PB explanations were used to explain why 0/4=0/12 than for any other example, perhaps indicating the need for students to relate zero to something concrete in order to comprehend this equivalence.

	Equal to the same number	Algorithmic	N/D	Zero is nothing	PB	Other
2/4=6/12	17 (35)	19 (40)	4 (8)	-	1 (2)	7 (15)
5/10=10/30	8 (16)	25 (52)	8 (16)	-	1 (2)	6 (13)
0/4=0/12	12 (24)	14 (29)	-	5 (10)	6 (13)	11 (23)

Table 7: Frequencies (in %) of types of explanations per pair of fractions (N=48)

With regard to consistency, 32 valid explanations were considered (see Table 8). Surprisingly, few students explained why 2/4=6/12 in the same way as they explained why 5/15=10/30. Also note, the consistency rate for all three examples of equivalent functions (19%) was similar to the consistency rate for all three examples of even numbers (17%).

Examples	2/4=6/12, 5/15=10/30	2/4=6/12, 0/4=0/12	5/15=10/30, 0/4=0/12	2/4=6/12, 5/15=10/30, 0/4=0/12
Consistent explanations	8 (25)	15 (43)	17 (50)	6 (19)

Table 8: Frequencies (in %) of consistent explanations for equivalent fractions (N=32)

Comparing the contexts

Finally, when considering both mathematical contexts, 32 students correctly evaluated all of the tasks and offered valid explanations. Out of those students, only four (12%) were consistent in the types of explanations they gave for both contexts, giving the same type of explanation to explain the parity of the three even numbers and the same type of explanation when explaining why the three pairs of fractions were equivalent.

SUMMARY AND DISCUSSION

Do different examples give rise to different explanations, or are some explanations more prevalent than others, regardless of the different examples? The answer to this question may be dependent on the context. With regard to parity, different examples elicited different explanations. Within the context of equivalent fractions, nearly half of all the fractions explanations were algorithmic. While algorithmic knowledge is an essential element of mathematics knowledge (Fischbein, 1993), this type of

explanation sheds little light on students' conceptualization of fractions and, specifically, equivalent fractions. When explaining why 2/4=6/12, over a third of the students pointed out that both of these fractions were equal to 1/2 and therefore equal to each other. Perhaps because this example is intuitive and close to the student's world, it was helpful in bringing out this explanation. This explanation focuses on the conceptualization of equivalent fractions as being different representation for the same quantity. On the other hand, the N/D explanation focuses on the conceptualization of a fraction as a ratio. Interestingly, this explanation hardly came up. Perhaps a different example, such as 3/9=5/15 would have elicited this explanation. Perhaps, in line with Watson and Mason's (2006) theory of task variation, different tasks might expose different relationships between examples and explanations.

Do students consistently give the same explanation for a concept despite being shown different examples of that concept or do they give different explanations for different examples? For both contexts, the general answer to this question was no, most students do not give the same explanation for different examples. However, upon a closer look, on the parity questionnaire, most students were consistent, not only when explaining the parity of 14 and 9, two relatively small numbers, but also when explaining the parity of 14 and 286. It was the introduction of zero, which caused students to think of other ways of explaining the parity of this number. On the fractions questionnaire, the opposite occurred. More research is needed in order to discern which examples will lead students to seek different explanations and which examples will lead students to use which explanations. Such research could be helpful to teachers planning lessons, as well as for researchers planning studies which involve the use of examples.

Mathematically, there is no reason to use different explanations when explaining why various instances of some concept are all examples of that concept. In fact, consistently using the same explanation may be seen as a sign of mathematical maturity. This study showed, however, that most students, at least during their younger years, do not consistently use the same explanation for each example. Zodik and Zaslavsky's (2008) study noted that teachers consider several issues when choosing examples to present in class. However, the types of explanations that different examples may elicit from students, was not considered. Students may need assistance in recognizing the generality of instances and in finding relationships and connections between examples. Recall that explanations may be used to help make mathematical connections clear (Hemmi, Lepik, & Viholainen, 2013) and deepen students' understanding (Nunokawa, 2010). Considering that different explanations may be based on different ways of conceptualizing a concept and may emphasize different attributes of a concept, teachers, as well as researchers, may consider the combination and integration of examples and explanations, when choosing which examples, and which explanations, to present to students in class.

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